# My Notes and Viewpoints on Complex and Real

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## I. Relationship between complex and real derivations

To begin with, in this manuscript, derivation of a real function with respect to real variable is named as *real derivation*, and derivation of a complex function with respect to complex variable is named as *complex derivation*.

### A. Scalar case

Define a complex variable $z = x + i \cdot y \in \mathbb{C}$, where $i$ stands for the imaginary unit, and $x, y \in \mathbb{R}$. Then the conjugate of variable $z$ is a new variable as $z^* = x - i \cdot y$. Here, $z$ and $z^*$ can be viewed as functions with respect to $x$ and $y$, and then we have the following complex derivative operators:

$$
\begin{aligned}
\frac{\partial}{\partial z} &= \frac{1}{2} \cdot 2 \cdot \frac{\partial}{\partial z} \\
&= \frac{1}{2} \cdot \left( \frac{\partial}{\partial z} + \frac{\partial}{\partial z} \right) \\
&= \frac{1}{2} \cdot \left( \frac{\partial}{\partial x} \cdot \frac{\partial x}{\partial z} + \frac{\partial}{\partial y} \cdot \frac{\partial y}{\partial z} \right) \\
&= \frac{1}{2} \cdot \left( \frac{\partial}{\partial x} \cdot \frac{1}{\frac{\partial z}{\partial x}} + \frac{\partial}{\partial y} \cdot \frac{1}{\frac{\partial z}{\partial y}} \right) \\
&= \frac{1}{2} \cdot \left( \frac{\partial}{\partial x} \cdot 1 + \frac{\partial}{\partial y} \cdot \frac{1}{i} \right) \\
&= \frac{1}{2} \cdot \left( \frac{\partial}{\partial x} - i \cdot \frac{\partial}{\partial y} \right)
\end{aligned} \tag{1a}
$$

$$
\frac{\partial}{\partial z^*} = \frac{1}{2} \cdot \left( \frac{\partial}{\partial x} + i \cdot \frac{\partial}{\partial y} \right). \tag{1b}
$$

### B. Vector case

In what follows, we use Eq. (1) and the definition of vector derivation to deduce the (complex) vector derivative operators.

Define a column vector variable $\mathbf{z} = \mathbf{x} + i \cdot \mathbf{y}$, where $\mathbf{z} = [z_1, z_2, \cdots, z_p]^T \in \mathbb{C}^p$, $\mathbf{x} = [x_1, x_2, \cdots, x_p]^T \in \mathbb{R}^p$ and $\mathbf{y} = [y_1, y_2, \cdots, y_p]^T \in \mathbb{R}^p$. According to the definition of a vector variable, we have

$$
\frac{\partial}{\partial \mathbf{z}} \triangleq \left[ \frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}, \cdots, \frac{\partial}{\partial z_p} \right]^T. \tag{2}
$$

Based on (1a), (2) can be reformed as

$$
\begin{aligned}
\frac{\partial}{\partial \mathbf{z}} &= \left[ \frac{1}{2} \cdot \left( \frac{\partial}{\partial x_1} - i \cdot \frac{\partial}{\partial y_1} \right) \ , \ \frac{1}{2} \cdot \left( \frac{\partial}{\partial x_2} - i \cdot \frac{\partial}{\partial y_2} \right) \ , \ \cdots \ , \ \frac{1}{2} \cdot \left( \frac{\partial}{\partial x_p} - i \cdot \frac{\partial}{\partial y_p} \right) \right]^T \\
&= \frac{1}{2} \cdot \left( \left[ \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \cdots, \frac{\partial}{\partial x_p} \right]^T - i \cdot \left[ \frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}, \cdots, \frac{\partial}{\partial y_p} \right]^T \right) \\
&= \frac{1}{2} \cdot \left( \frac{\partial}{\partial \mathbf{x}} - i \cdot \frac{\partial}{\partial \mathbf{y}} \right).
\end{aligned} \tag{3a}
$$

Similarly, we have

$$
\frac{\partial}{\partial \mathbf{z}^*} = \frac{1}{2} \cdot \left( \frac{\partial}{\partial \mathbf{x}} + i \cdot \frac{\partial}{\partial \mathbf{y}} \right). \tag{3b}
$$

### C. An example

In this section, we deduce the derivation of $\mathbf{w}^H\mathbf{R}\mathbf{w}$ with respect to $\mathbf{w}$. Here, $\mathbf{R}$ is a Hermitian matrix. $\mathbf{w}^H\mathbf{R}\mathbf{w}$ can be written as

$$\begin{aligned}\mathbf{w}^H\mathbf{R}\mathbf{w} =& \left(\mathbf{w}_R + i\cdot\mathbf{w}_I\right)^H\left(\mathbf{R}_R + i\cdot\mathbf{R}_I\right)\left(\mathbf{w}_R + i\cdot\mathbf{w}_I\right)\\ =&\mathbf{w}_R^T\mathbf{R}_R\mathbf{w}_R - \mathbf{w}_R^T\mathbf{R}_I\mathbf{w}_I + \mathbf{w}_I^T\mathbf{R}_R\mathbf{w}_I + \mathbf{w}_I^T\mathbf{R}_I\mathbf{w}_R\\ &+ i\cdot\left(\mathbf{w}_R^T\mathbf{R}_R\mathbf{w}_I + \mathbf{w}_R^T\mathbf{R}_I\mathbf{w}_R - \mathbf{w}_I^T\mathbf{R}_R\mathbf{w}_R + \mathbf{w}_I^T\mathbf{R}_I\mathbf{w}_I\right)\\ =&\mathbf{w}_R^T\mathbf{R}_R\mathbf{w}_R - \mathbf{w}_R^T\mathbf{R}_I\mathbf{w}_I + \mathbf{w}_I^T\mathbf{R}_R\mathbf{w}_I + \mathbf{w}_I^T\mathbf{R}_I\mathbf{w}_R\end{aligned} \tag{4}$$

where $\mathbf{w}_R$ and $\mathbf{R}_R$ are the real part of $\mathbf{w}$ and $\mathbf{R}$, respectively, while $\mathbf{w}_I$ and $\mathbf{R}_I$ are the imaginary part of $\mathbf{w}$ and $\mathbf{R}$, respectively, i.e., $\mathbf{w} \triangleq \mathbf{w}_R + i\cdot\mathbf{w}_I$ and $\mathbf{R} \triangleq \mathbf{R}_R + i\cdot\mathbf{R}_I$. The last equation in (4) holds because the imaginary part of $\mathbf{w}^H\mathbf{R}\mathbf{w}$ is 0 since $\mathbf{R}$ is a positive semi-definite matrix ($\mathbf{R}$ is Hermitian).

According to (3a), we can obtain

$$\begin{aligned}\frac{\partial\mathbf{w}^H\mathbf{R}\mathbf{w}}{\partial\mathbf{w}} =&\frac{1}{2}\cdot\left(\frac{\partial\mathbf{w}^H\mathbf{R}\mathbf{w}}{\partial\mathbf{w}_R} - i\cdot\frac{\partial\mathbf{w}^H\mathbf{R}\mathbf{w}}{\partial\mathbf{w}_I}\right)\\ =&\frac{1}{2}\cdot\left[2\mathbf{R}_R\mathbf{w}_R - \mathbf{R}_I\mathbf{w}_I + \mathbf{R}_I^T\mathbf{w}_I - i\cdot\left(-\mathbf{R}_I^T\mathbf{w}_R + 2\mathbf{R}_R\mathbf{w}_I + \mathbf{R}_I\mathbf{w}_R\right)\right]\\ =&\frac{1}{2}\cdot\left[2\mathbf{R}_R\mathbf{w}_R - 2\mathbf{R}_I\mathbf{w}_I - i\cdot\left(2\mathbf{R}_I\mathbf{w}_R + 2\mathbf{R}_R\mathbf{w}_I\right)\right]\\ =&\mathbf{R}_R\mathbf{w}_R - \mathbf{R}_I\mathbf{w}_I - i\mathbf{R}_I\mathbf{w}_R - i\mathbf{R}_R\mathbf{w}_I\\ =&(\mathbf{R}_R - i\mathbf{R}_I)(\mathbf{w}_R - i\mathbf{w}_I)\\ =&(\mathbf{R}_R^T + i\mathbf{R}_I^T)(\mathbf{w}_R - i\mathbf{w}_I)\\ =&\mathbf{R}^T\mathbf{w}^*\end{aligned} \tag{5}$$

where $\mathbf{R}_R^T = \mathbf{R}_R$ and $\mathbf{R}_I^T = -\mathbf{R}_I$ (since $\mathbf{R}$ is a Hermitian matrix, i.e., $\mathbf{R}^H = \mathbf{R}$) are adopted in the third and sixth equations.

## II. Transfer from complex operation to real operation

In many references, some complex operations are almost always translated into the real ones, such as $\mathbf{w}^H\mathbf{R}\mathbf{w}$. Specifically speaking, given $\mathbf{w} \in \mathbb{C}^{M\times 1}$, $\mathbf{R} \in \mathbb{C}^{M\times M}$ and $\mathbf{R}$ is a Hermitian matrix, we can calculate $\mathbf{w}^H\mathbf{R}\mathbf{w}$ using

$$\mathbf{w}^H\mathbf{R}\mathbf{w} = \begin{bmatrix}\mathbf{w}_R\\ \mathbf{w}_I\end{bmatrix}\begin{bmatrix}\mathbf{R}_R & -\mathbf{R}_I\\ \mathbf{R}_I & \mathbf{R}_R\end{bmatrix}\begin{bmatrix}\mathbf{w}_R\\ \mathbf{w}_I\end{bmatrix} \tag{6}$$

where $\mathbf{w} \triangleq \mathbf{w}_R + i\cdot\mathbf{w}_I$ and $\mathbf{R} \triangleq \mathbf{R}_R + i\cdot\mathbf{R}_I$.

The proof of (6) is provided as follows. Firstly, the left side of (6) can be written as

$$\begin{aligned}\mathbf{w}^H\mathbf{R}\mathbf{w} =& \left[\mathbf{w}_R + i\cdot\mathbf{w}_I\right]^H\left[\mathbf{R}_R + i\cdot\mathbf{R}_I\right]\left[\mathbf{w}_R + i\cdot\mathbf{w}_I\right]\\ =& \left[\mathbf{w}_R^T - i\cdot\mathbf{w}_I^T\right]\left[\mathbf{R}_R + i\cdot\mathbf{R}_I\right]\left[\mathbf{w}_R + i\cdot\mathbf{w}_I\right]\\ =&\mathbf{w}_R^T\mathbf{R}_R\mathbf{w}_R + \mathbf{w}_R^T\mathbf{R}_R i\mathbf{w}_I + \mathbf{w}_R^T i\mathbf{R}_I\mathbf{w}_R + \mathbf{w}_R^T i\mathbf{R}_I i\mathbf{w}_I\\ &- i\mathbf{w}_I^T\mathbf{R}_R\mathbf{w}_R - i\mathbf{w}_I^T\mathbf{R}_R i\mathbf{w}_I - i\mathbf{w}_I^T i\mathbf{R}_I\mathbf{w}_R - i\mathbf{w}_I^T i\mathbf{R}_I i\mathbf{w}_I\end{aligned} \tag{7}$$

in which, the sum of the second and the fifth terms equals to zero, i.e., $\mathbf{w}_R^T\mathbf{R}_R i\mathbf{w}_I - i\mathbf{w}_I^T\mathbf{R}_R\mathbf{w}_R = 0$. Since $\mathbf{R}$ is a Hermitian matrix, $\mathbf{R}_I$ has the form as

$$\mathbf{R}_I = \begin{bmatrix}0 & r_{12} & \cdots & r_{1M}\\ -r_{12} & 0 & \cdots & r_{2M}\\ \vdots & \vdots & \ddots & \vdots\\ -r_{1M} & -r_{2M} & \cdots & 0\end{bmatrix} \tag{8}$$

and it is easy to find that the third and the eighth terms in (7) are both zero, i.e., $\mathbf{w}_R^T i\mathbf{R}_I\mathbf{w}_R = 0$ and $i\mathbf{w}_I^T i\mathbf{R}_I i\mathbf{w}_I = 0$. Therefore, (7) can be finally revised as

$$\begin{aligned}\mathbf{w}^H\mathbf{R}\mathbf{w} =&\mathbf{w}_R^T\mathbf{R}_R\mathbf{w}_R + \mathbf{w}_R^T i\mathbf{R}_I i\mathbf{w}_I - i\mathbf{w}_I^T\mathbf{R}_R i\mathbf{w}_I - i\mathbf{w}_I^T i\mathbf{R}_I\mathbf{w}_R\\ =&\mathbf{w}_R^T\mathbf{R}_R\mathbf{w}_R - \mathbf{w}_R^T\mathbf{R}_I\mathbf{w}_I + \mathbf{w}_I^T\mathbf{R}_R\mathbf{w}_I + \mathbf{w}_I^T\mathbf{R}_I\mathbf{w}_R.\end{aligned} \tag{9}$$

On the other hand, the right side of (6) can be written as

$$\begin{bmatrix}\mathbf{w}_R\\ \mathbf{w}_I\end{bmatrix}\begin{bmatrix}\mathbf{R}_R & -\mathbf{R}_I\\ \mathbf{R}_I & \mathbf{R}\end{bmatrix}\begin{bmatrix}\mathbf{w}_R\\ \mathbf{w}_I\end{bmatrix} = \mathbf{w}_R^T\mathbf{R}_R\mathbf{w}_R + \mathbf{w}_I^T\mathbf{R}_I\mathbf{w}_R - \mathbf{w}_R^T\mathbf{R}_I\mathbf{w}_I + \mathbf{w}_I^T\mathbf{R}_R\mathbf{w}_I. \tag{10}$$

By comparing (9) with (10), we obtain (6).