The Difference Between Capon and MUSIC Algorithms

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Abstract

In this report, the Capon and MUSIC algorithms are reviewed, and the difference between these two algorithms are analysed. Two conclusions are obtained: On one hand, if the SNR is large enough, the spectrums of Capon and MUSIC are approximately the same, and hence their performances may be similar. On the other hand, MUSIC algorithm performs better than Capon algorithm when the separation angle of sources is quite small, and this is why MUSIC (or saying subspace-based methods) is called as high-resolution algorithm.

Index Terms

Capon algorithm, MUSIC algorithm, direction-of-arrival estimation

I. Introduction

IN this report, the difference between Capon and MUSIC algorithms are analysed. These two algorithms are first reviewed in Sections II and III, respectively. My personal viewpoints of the difference between these two methods is reported in Section IV. Section V presents some simulation results. Two conclusions are proposed in Sections IV and V, respectively.

II. Capon Algorithm

Consider a data model as:

$$\mathbf{x}(t) = \mathbf{A}\mathbf{s}(t) + \mathbf{n}(t) \tag{1}$$

where $\mathbf{x}(t)$ is observation data vector, $\mathbf{A}$ is so-called steering matrix in array signal processing, $\mathbf{s}(t)$ and $\mathbf{n}(t)$ stand for signal and noise vectors, respectively, and t denotes time index. Put a weight vector $\mathbf{w}$ onto the observation vector $\mathbf{x}(t)$, and we get the output as

$$y(t) = \mathbf{w}^H \mathbf{x}(t). \tag{2}$$

Therefore, the power of the array output can be formulated as follows

$$R_y = \mathrm{E}\{|y(t)|^2\} = \mathbf{w}^H \mathbf{R}_{\mathbf{x}} \mathbf{w} \tag{3}$$

where $\mathrm{E}\{\cdot\}$ and $\cdot^H$ denote mathematical expectation and Hermitian transpose, respectively. In addition, $\mathbf{R}_{\mathbf{x}} = \mathrm{E}\{\mathbf{x}(t)\mathbf{x}^H(t)\}$ is the covariance matrix of the observation data.

The Capon algorithm [1] can be described as: minimize the output power, while maintaining a unity gain in look direction. It can be formulated as follows

$$\begin{aligned} \min_{\mathbf{w}} \quad & \mathbf{w}^H \mathbf{R}_{\mathbf{x}} \mathbf{w} \\ \text{subject to} \quad & \mathbf{w}^H \mathbf{a}(\theta) = 1. \end{aligned} \tag{4}$$

Equation (4) can be solved using Lagrange multiplier method and its solution is

$$\mathbf{w}_{Lag} = \frac{\mathbf{R}_{\mathbf{x}}{}^{-1} \mathbf{a}(\theta)}{\mathbf{a}^H(\theta) \mathbf{R}_{\mathbf{x}}{}^{-1} \mathbf{a}(\theta)}. \tag{5}$$

Substituting $\mathbf{w}_{Lag}$ back into Equation (3), one can get the output power associated with directions as

$$P_{\text{Capon}}(\theta) = \frac{1}{\mathbf{a}^H(\theta) \mathbf{R}_{\mathbf{x}}{}^{-1} \mathbf{a}(\theta)}. \tag{6}$$

We can use $P_{\text{Capon}}(\theta)$ to search for the directions-of-arrival of the incoming sources, by computing and plot the spatial spectrum in the whole spatial range of θ.

III. MUSIC ALGORITHM

Once we get the covariance matrix of the observation data, $\mathbf{R_x}$, we perform the eigenvalue decomposition on it and obtain the signal and noise components, as

$$\begin{aligned}\mathbf{R_x} &= \mathbf{U_s}\mathbf{\Sigma_s}\mathbf{U_s}^H + \mathbf{U_n}\mathbf{\Sigma_n}\mathbf{U_n}^H \\ &= \sum \sigma_s \mathbf{u_s}\mathbf{u_s}^H + \sum \sigma_n \mathbf{u_n}\mathbf{u_n}^H.\end{aligned} \tag{7}$$

According to the orthogonality between the signal and noise subspaces [2], we can form the MUSIC spatial spectrum as follows

$$P_{\text{MUSIC}}(\theta) = \frac{1}{\mathbf{a}^H(\theta)\mathbf{U_n}\mathbf{U_n}^H\mathbf{a}(\theta)}. \tag{8}$$

IV. MY ANALYSE

In this section, we analyse the difference between the Capon and MUSIC algorithms, using Equations (6), (7) and (8). It is easy to find that $\mathbf{R_x}^{-1}$ in Equation (6) can be written as

$$\begin{aligned}\mathbf{R_x}^{-1} &= (\mathbf{U_s}\mathbf{\Sigma_s}\mathbf{U_s}^H + \mathbf{U_n}\mathbf{\Sigma_n}\mathbf{U_n}^H)^{-1} \\ &= \mathbf{U_s}\mathbf{\Sigma_s}^{-1}\mathbf{U_s}^H + \mathbf{U_n}\mathbf{\Sigma_n}^{-1}\mathbf{U_n}^H \\ &= \sum \frac{1}{\sigma_s}\mathbf{u_s}\mathbf{u_s}^H + \sum \frac{1}{\sigma_n}\mathbf{u_n}\mathbf{u_n}^H.\end{aligned} \tag{9}$$

If the signal-to-noise ratio (SNR) is large enough, i.e., $\frac{\sigma_s}{\sigma_n}$ is large enough, then the components $\frac{1}{\sigma_s}\mathbf{u_s}\mathbf{u_s}^H$ can be neglected compared to $\frac{1}{\sigma_n}\mathbf{u_n}\mathbf{u_n}^H$. Therefore, in this case (SNR is large), Equation (6) can be approximately rewritten as

$$P_{\text{Capon}}(\theta) \simeq \frac{1}{\mathbf{a}^H(\theta)\mathbf{U_n}\mathbf{\Sigma_n}^{-1}\mathbf{U_n}^H\mathbf{a}(\theta)}. \tag{10}$$

Because $\mathbf{\Sigma_n}^{-1}$ does not change the spectrum, $P_{\text{Capon}}(\theta)$ can be further written as

$$P_{\text{Capon}}(\theta) \simeq \frac{1}{\mathbf{a}^H(\theta)\mathbf{U_n}\mathbf{U_n}^H\mathbf{a}(\theta)} = P_{\text{MUSIC}}(\theta). \tag{11}$$

Therefore, we come to the conclusion that: **If the SNR is large enough, the spectrums of Capon and MUSIC are approximately the same, and hence their performances may be similar.** This conclusion can be verified in simulation in the next section.

V. SIMULATION

Example 1: Consider two sources with look directions of $\{-10°, 20°\}$, 8 sensors (uniform linear array), 500 snapshots, and SNR ranges from -10dB to 20dB. 1000 Mente-Carlo trials are run. The root-mean-squared error (RMSE) versus SNR of Capon as well as MUSIC algorithms are drawn in Fig. 1. From Fig. 1, we can achieve the conclusion presented in Section IV.

Example 2: This experiment is designed to see what may happen if the separation angle between two incident signals gets smaller. Consider two sources, 8 sensors (uniform linear array), 500 snapshots, and SNR is set to be 10dB. 1000 Mente-Carlo trials are run. The look direction of the first source is chosen from the angle set of $\{0°, 5°, 10°, 15°, 16°, 17°, 18°\}$, while the direction of the second source is fixed at $20°$. The results of RMSE versus separation angle are plotted in Fig. 2, from which we can obtain the conclusion that: **MUSIC algorithm performs better than Capon algorithm when the separation angle of sources is quite small, and this is why MUSIC (or saying subspace-based methods) is called as high-resolution algorithm.**

REFERENCES

[1] H. Krim and M. Viberg, "Two decades of array signal processing research: the parametric approach," *IEEE Signal Processing Magazine*, vol. 13, no. 4, pp. 67-94, July 1996.

[2] R. Schmidt, "Multiple emitter location and signal parameter estimation," *IEEE Transactions on Antennas and Propagation*, vol. 34, no. 3, pp. 276-280, March 1986.

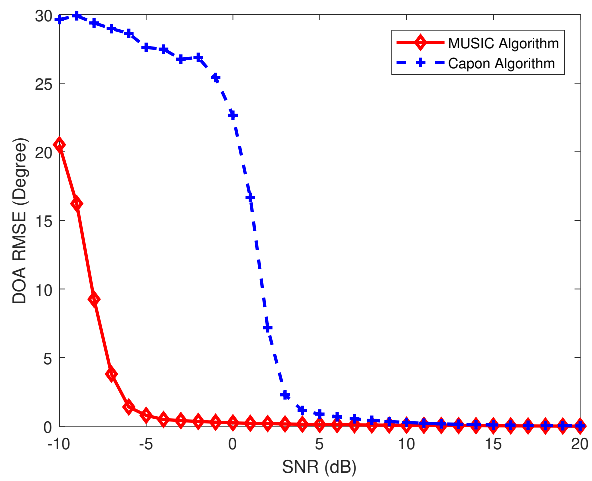

Fig. 1. RMSE versus SNR of Capon and MUSIC algorithms, with two incident signals from $\{-10°, 20°\}$.

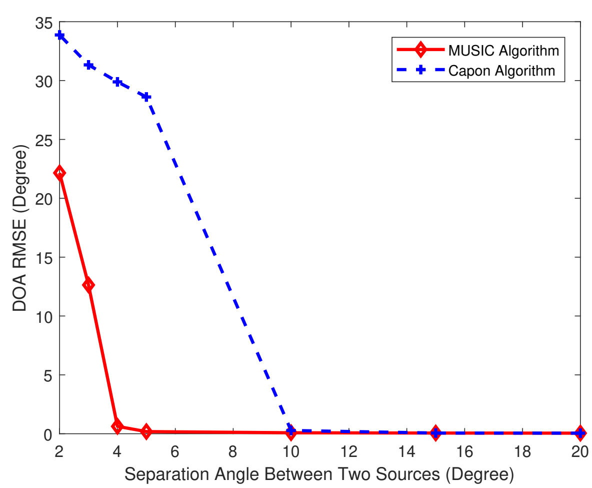

Fig. 2. RMSE versus separation angle of sources of Capon and MUSIC algorithms, with $\mathrm{SNR} = 10\mathrm{dB}$.